(Weighted) Dyadic Maximal Function

We will often make use of the dyadic maximal function ($\mathbb{R}^n$ equipped w/ dyadic grid $\mathcal{D}$):

$$M_{\mathcal{D}} f(x) := \sup_{Q \in \mathcal{D}} \left(\langle |f| \rangle_Q \mathbb{1}_Q(x) \right) = \sup_{Q \ni x} \langle |f| \rangle_Q$$

(the discrete, dyadic version of the classical, Hardy-Littlewood maximal function).
Since the generalization is very simple & useful, we work instead with:

Let μ be a <u>locally finite</u> measure on $\mathbb{R}^n$ (i.e. $\mu(Q) < \infty$, $\forall$ cube Q)
and $\mathcal{D}$ be a dyadic grid on $\mathbb{R}^n$. The dyadic maximal function wrt μ and $\mathcal{D}$:

$$\boxed{M_{\mathcal{D}}^{\mu} f(x) := \sup_{Q \in \mathcal{D}} \left(\mathbb{E}_Q^{\mu} |f| \cdot \mathbb{1}_Q(x) \right)} \quad \text{where} \quad \boxed{\mathbb{E}_Q^{\mu} f := \frac{1}{\mu(Q)} \int_Q f \, d\mu}$$

Thm.: (Weak 1,1 for $M_{\mathcal{D}}^{\mu}$)

The weighted dyadic maximal function $M_{\mathcal{D}}^{\mu}$ is bounded as an operator $M_{\mathcal{D}}^{\mu} : L^1(\mu) \to L^{1,\infty}(\mu)$
where $\|f\|_{L^{1,\infty}(\mu)} := \sup_{\lambda > 0} \lambda \cdot \mu\{|f| > \lambda\}$. In fact, more is true:

$$\boxed{\lambda \cdot \mu\{ M_{\mathcal{D}}^{\mu} f > \lambda \} \leq \int_{\{M_{\mathcal{D}}^{\mu} f > \lambda\}} |f| \, d\mu} \leq \|f\|_{L^1(\mu)}$$

<u>Proof</u>: Let $\mathcal{F}_\lambda := \{ Q \in \mathcal{D} : \mathbb{E}_Q^{\mu} |f| > \lambda \}$. Then:

$$\boxed{\{ M_{\mathcal{D}}^{\mu} f > \lambda \} = \bigcup_{Q \in \mathcal{F}_\lambda} Q}$$

$x \in Q, Q \in \mathcal{F}_\lambda \Rightarrow \mathbb{E}_Q^{\mu} |f| > \lambda$ and $x \in Q$
$\Rightarrow M_{\mathcal{D}}^{\mu} f(x) \geq \mathbb{E}_Q^{\mu} |f| > \lambda \ldots$
$\Rightarrow \bigcup_{Q \in \mathcal{F}_\lambda} Q \subseteq \{ M_{\mathcal{D}}^{\mu} f > \lambda \}$.

$M_{\mathcal{D}}^{\mu} f(x) > \lambda \Rightarrow \exists Q \in \mathcal{D}$ s.t. $x \in Q$ & $\mathbb{E}_Q^{\mu} |f| > \lambda$
$\Rightarrow x \in Q$ & $Q \in \mathcal{F}_\lambda \Rightarrow \{ M_{\mathcal{D}}^{\mu} f > \lambda \} \subseteq \bigcup_{Q \in \mathcal{F}_\lambda} Q$

Let $\mathcal{F} \subset \mathcal{F}_\lambda$ be any <u>finite subcollection</u> of $\mathcal{F}_\lambda$, and
let $\mathcal{F}^*$ be the set of <u>maximal</u> cubes in $\mathcal{F}$: $\mathcal{F}^* := \{ Q \in \mathcal{F} : \nexists Q' \in \mathcal{F} \text{ s.t. } Q \subsetneq Q' \}$.

<u>Claim</u>: Every $Q \in \mathcal{F}$ is contained in some maximal $Q^* \in \mathcal{F}^*$.

If Q is maximal, we are done. Otherwise, $\exists Q_1 \in \mathcal{F}$ s.t. $Q \subsetneq Q_1$. If Q_1 is maximal, we are done.
Otherwise, $\exists Q_2 \in \mathcal{F}$ s.t. $Q \subsetneq Q_1 \subsetneq Q_2 \ldots$
Since $\mathcal{F}$ = finite collection, this process must terminate, i.e. $Q \subsetneq Q_1 \subsetneq \ldots \subsetneq Q_n$
and Q_n is not contained in any other element of $\mathcal{F}$, i.e. Q_n is maximal. ∎

$\Rightarrow \boxed{\bigcup_{Q \in \mathcal{F}} Q = \bigcup_{Q \in \mathcal{F}^*} Q}$ Moreover: the latter is a <u>disjoint</u> union, i.e. the elements of $\mathcal{F}^*$ are <u>disjoint</u>.

$Q^*, P^* \in \mathcal{F}^*$; $Q^* \cap P^* \neq \emptyset \Rightarrow Q^* \subsetneq P^*$ or $P^* \subsetneq Q^* \Rightarrow$ contradiction $\Rightarrow P^* = Q^*$

$$\Rightarrow \mu\left(\bigcup_{Q \in \mathcal{F}} Q\right) = \mu\left(\bigcup_{Q^* \in \mathcal{F}^*} Q^*\right) = \sum_{Q^* \in \mathcal{F}^*} \mu(Q^*) \leq \sum_{Q^* \in \mathcal{F}^*} \frac{1}{\lambda} \int_{Q^*} |f| \, d\mu = \frac{1}{\lambda} \int_{\bigcup_{Q^* \in \mathcal{F}^*} Q^*} |f| \, d\mu \leq \frac{1}{\lambda} \int_{\{M_{\mathcal{D}}^{\mu} f > \lambda\}} |f| \, d\mu$$

$$\left[Q^* \in \mathcal{F}^* \subset \mathcal{F} \Rightarrow \mathbb{E}_Q^{\mu} |f| = \frac{1}{\mu(Q)} \int_Q |f| \, d\mu > \lambda \right]$$

Finally: let $\mathcal{F}_n$ be an increasing sequence of finite subcollections of $\mathcal{F}_\lambda$ s.t. $\bigcup_{n=1}^{\infty} \mathcal{F}_n = \mathcal{F}_\lambda$
(since $\mathcal{F}_\lambda \subset \mathcal{D}$, it is countable)

$$\Rightarrow \mu\{ M_{\mathcal{D}}^{\mu} f > \lambda \} = \mu\left(\bigcup_{Q \in \mathcal{F}_\lambda} Q \right) = \lim_{n \to \infty} \mu\left(\bigcup_{Q \in \mathcal{F}_n} Q \right) \leq \frac{1}{\lambda} \int_{\{M_{\mathcal{D}}^{\mu} f > \lambda\}} |f| \, d\mu.$$

∎

Corollary (Strong p,p for $M^\mu_{\mathcal{D}}$)
For all $1<p<\infty$, there holds:

$$\boxed{\| M^\mu_{\mathcal{D}} : L^p(\mu) \to L^p(\mu) \| \leq p' = \frac{p}{p-1}}$$

(X,μ) measure space; $\lambda \geq 0$; $\varphi : [\lambda,\infty) \to [0,\infty)$ increasing, continuously diff'ble, $\varphi(\lambda)=0$;

$$\int_{\{x\in X: |f(x)|>\lambda\}} \varphi(|f(x)|)\,d\mu(x) = \int_\lambda^\infty \varphi'(t)\,\mu\{x\in X : |f(x)|>t\}\,dt$$

Proof:

$$\|M^\mu_{\mathcal{D}} f\|^p_{L^p(\mu)} = \int_0^\infty p t^{p-1} \mu\{M^\mu_{\mathcal{D}} f > t\}\,dt$$

$$\leq \int_0^\infty p t^{p-1} \frac{1}{t} \int_{\{M^\mu_{\mathcal{D}} f>t\}} |f|\,d\mu\,dt$$

$$= \int_{\mathbb{R}^n} |f(x)| \underbrace{\left(\int_0^{M^\mu_{\mathcal{D}} f(x)} p t^{p-2}\,dt \right)}_{= p\frac{t^{p-1}}{p-1}\Big|_0^{M^\mu_{\mathcal{D}} f(x)} = \frac{p}{p-1}\left(M^\mu_{\mathcal{D}} f(x)\right)^{p-1}} d\mu(x)$$

$$= \int_{\mathbb{R}^n} |f(x)| \frac{p}{p-1} \left(M^\mu_{\mathcal{D}} f(x)\right)^{p-1} d\mu(x)$$

$$\leq p' \left(\int_{\mathbb{R}^n} |f(x)|^p d\mu(x) \right)^{1/p} \left(\int_{\mathbb{R}^n} \left(M^\mu_{\mathcal{D}} f(x)\right)^{\overbrace{p'(p-1)}^{=p}} d\mu(x) \right)^{\frac{1}{p'}}$$

$$= p' \|f\|_{L^p(\mu)} \|M^\mu_{\mathcal{D}} f\|^{p/p'}_{L^p(\mu)} = p' \|f\|_{L^p(\mu)} \|M^\mu_{\mathcal{D}} f\|^{p-1}_{L^p(\mu)}$$

$$\Rightarrow \|M^\mu_{\mathcal{D}} f\|^p_{L^p(\mu)} \leq p' \|f\|_{L^p(\mu)} \|M^\mu_{\mathcal{D}} f\|^{p-1}_{L^p(\mu)} \quad (*)$$

$$\Rightarrow \boxed{\|M^\mu_{\mathcal{D}} f\|_{L^p(\mu)} \leq p' \|f\|_{L^p(\mu)}}$$

Remark: In $(*)$ we divide by $\|M^\mu_{\mathcal{D}} f\|_{L^p(\mu)}$, assuming $\|M^\mu_{\mathcal{D}} f\|_{L^p(\mu)} < \infty$.
To guarantee finiteness: first run the computation w/ a maximal operator defined by a finite subcollection $\mathcal{F} \subset \mathcal{D}$, to conclude that:

$$\|M^\mu_{\mathcal{F}} f\|_{L^p(\mu)} \leq p' \|f\|_{L^p(\mu)}$$

Then apply MCT to an increasing family $\mathcal{F}_n$ s.t. $\mathcal{D} = \bigcup_{n=1}^\infty \mathcal{F}_n$, to get the general result

$$\|M^\mu_{\mathcal{D}} f\|_{L^p(\mu)} = \lim_{n\to\infty} \|M^\mu_{\mathcal{F}_n} f\|_{L^p(\mu)} \leq p' \|f\|_{L^p(\mu)}.$$

∎

§. The Sharp Function

Def.: For a function $f \in L^1_{loc}(\mathbb{R})$, define its sharp function $f^{\#}$ by:

$$\boxed{f^{\#}(x) := \sup_{I:\,x\in I} \frac{1}{|I|}\int_I |f(y)-\langle f\rangle_I|\,dy}$$

where the supremum is over all intervals I containing x. Given a dyadic grid $\mathcal{D}$ on $\mathbb{R}$, one can also define a dyadic version of $f^{\#}$:

$$\boxed{f^{\#}_{\mathcal{D}}(x) := \sup_{\substack{I\in\mathcal{D}\\ x\in I}} \frac{1}{|I|}\int_I |f(y)-\langle f\rangle_I|\,dy}$$

→ This is a variant of the maximal function, focusing not on the values of f but on its deviation from local mean values.

→ Can be used to describe $BMO(\mathbb{R})$ – or $BMO_{\mathcal{D}}(\mathbb{R})$: the space of functions of bounded mean oscillation

$$\|b\|_{BMO} = \sup_I \frac{1}{|I|}\int_I |b-\langle b\rangle_I|$$

→ $\forall x\in\mathbb{R}$: $\forall I \ni x$, $\frac{1}{|I|}\int_I |b-\langle b\rangle_I| \le \|b\|_{BMO} \Rightarrow b^{\#}(x) \le \|b\|_{BMO}, \forall x\in\mathbb{R} \Rightarrow \|b^{\#}\|_\infty \le \|b\|_{BMO}$

→ For any interval I: $\frac{1}{|I|}\int_I |b-\langle b\rangle_I| \le b^{\#}(x) \le \|b^{\#}\|_\infty$ (for some $x\in I$) $\Rightarrow \|b\|_{BMO} \le \|b^{\#}\|_\infty$.

$$\|b\|_{BMO} = \|b^{\#}\|_\infty \quad \text{and} \quad \|b\|_{BMO_{\mathcal{D}}} = \|b^{\#}_{\mathcal{D}}\|_\infty.$$

→ Plays a key role in interpolation w/ BMO:

$$\boxed{\begin{array}{l} T \text{ linear},\ 1\le p_0<\infty \\ \left.\begin{array}{l} T: L^{p_0}(\mathbb{R})\to L^{p_0}(\mathbb{R}) \\ T: L^\infty(\mathbb{R}) \to BMO(\mathbb{R}) \end{array}\right\} \Rightarrow T: L^p(\mathbb{R})\to L^p(\mathbb{R}),\ p_0\le p<\infty \end{array}}$$

→ Relation to maximal functions:

$$\frac{1}{|I|}\int_I |f-\langle f\rangle_I| \le 2\langle |f|\rangle_I$$

$$\Rightarrow \sup_{I:\,x\in I}\frac{1}{|I|}\int_I |f-\langle f\rangle_I| \le 2\sup_{I:\,x\in I}\langle |f|\rangle_I$$

$$\boxed{\begin{array}{l} f^{\#}(x) \le 2Mf(x) \\ f^{\#}_{\mathcal{D}}(x) \le 2M_{\mathcal{D}}f(x) \end{array}}$$

$$\Rightarrow \boxed{\begin{array}{l} \|f^{\#}\|_p \lesssim \|f\|_p,\ 1<p<\infty \\ \|f^{\#}_{\mathcal{D}}\|_p \lesssim \|f\|_p,\ 1<p<\infty \end{array}}$$

→ **Main point:** The reverse inequality holds, i.e.

$$\boxed{\begin{array}{l} \|f\|_p \lesssim \|f^{\#}\|_p,\ 1<p<\infty \\ \|f\|_p \lesssim \|f^{\#}_{\mathcal{D}}\|_p,\ 1<p<\infty \end{array}}$$

proof is a classic example of a "good-λ inequality" proof.

$$\Rightarrow \boxed{\|f^{\#}\|_p \simeq \|f\|_p \simeq \|f^{\#}_{\mathcal{D}}\|_p,\ 1<p<\infty}$$

→ Connection with our square function proof? It allows us to work instead with $S_{\mathcal{D}}^2 f$!
This is done via the following variation on the sharp function:

Def.: For a function $f \in L^1_{loc}(\mathbb{R})$ and $1<p<\infty$, define:

$$\boxed{f^{\#,p}(x) := \sup_{I:\, x \in I} \left(\frac{1}{|I|} \int_I |f(y) - \langle f \rangle_I|^p \, dy \right)^{1/p}}$$

and its obvious dyadic analogue $f_{\mathcal{D}}^{\#,p}$

We will see that this satisfies:

Theorem: Let $f \in L^1_{loc}(\mathbb{R})$ and $1<p_0<\infty$. Then:

→ $\boxed{\|f\|_p \lesssim \|f^{\#,p_0}\|_p}$ $\forall\, 1<p<\infty$

→ If $p>p_0$, the reverse inequality holds: $\boxed{\|f^{\#,p_0}\|_p \lesssim \|f\|_p}$ $\forall\, p>p_0$

All the same results hold for $f_{\mathcal{D}}^{\#,p}$.

For now, we prove the second part (this was the easy part when $p_0=1$).

$$\frac{1}{|I|}\int_I |f(y) - \langle f \rangle_I|^{p_0} dy \lesssim \frac{1}{|I|}\int_I \left(|f(y)|^{p_0} + |\langle f \rangle_I|^{p_0} \right) dy = \langle |f|^{p_0} \rangle_I + |\langle f \rangle_I|^{p_0} \leq 2 \langle |f|^{p_0} \rangle_I$$

$\Rightarrow f^{\#,p_0}(x) \lesssim \left(M(|f|^{p_0}) \right)^{1/p_0}(x)$

$$\Rightarrow \|f^{\#,p_0}\|_p \lesssim \|M(|f|^{p_0})^{1/p_0}\|_p = \left(\int M(|f|^{p_0})^{p/p_0} \right)^{1/p} = \|M(|f|^{p_0})\|_{p/p_0}^{1/p_0} \quad (p/p_0 > 1)$$

$$\lesssim \| |f|^{p_0} \|_{p/p_0}^{1/p_0} = \left(\int |f|^p \right)^{\frac{p_0}{p} \cdot \frac{1}{p_0}} = \|f\|_p.$$

§. Proof of $\|S_{\mathcal{D}} f\|_p \lesssim \|f\|_p$ for $p>2$, using the Sharp Function:

The bulk of the proof is the pointwise estimate $\boxed{(S_{\mathcal{D}}^2 f)_{\mathcal{D}}^{\#}(x) \leq 2\left(f_{\mathcal{D}}^{\#,2}\right)^2(x)}$ $(*)$

Once we have this:

$$\|S_{\mathcal{D}} f\|_p = \|S_{\mathcal{D}}^2 f\|_{p/2}^{1/2} \underset{p>2}{\lesssim} \|(S_{\mathcal{D}}^2 f)_{\mathcal{D}}^{\#}\|_{p/2}^{1/2} \lesssim \|(f_{\mathcal{D}}^{\#,2})^2\|_{p/2}^{1/2} = \|f^{\#,2}\|_p \underset{p>2}{\lesssim} \|f\|_p \qquad \underline{\text{Done}}$$

Proof of $(*)$: Fix $I \in \mathcal{D}$ for a minute and let $x \in I$. Then:

$$S_{\mathcal{D}}^2 f(x) = \sum_{J\in\mathcal{D}} (f,h_J)^2 \frac{\mathbb{1}_J(x)}{|J|} = \sum_{J\subset I} (f,h_J)^2 \frac{\mathbb{1}_J(x)}{|J|} + \sum_{J \supsetneq I} (f,h_J)^2 \frac{1}{|J|}$$

$\hookrightarrow$ these must intersect I

$$\Rightarrow \langle S_{\mathcal{D}}^2 f\rangle_I = \frac{1}{|I|}\int_I \sum_J (f,h_J)^2 \frac{\mathbb{1}_J(x)}{|J|}\,dx = \frac{1}{|I|}\sum_{J\in\mathcal{D}} (f,h_J)^2 \frac{|I\cap J|}{|J|}$$

$$= \frac{1}{|I|}\sum_{J\subset I}(f,h_J)^2 + \sum_{J\supsetneq I}(f,h_J)^2\frac{1}{|J|}$$

$$\Rightarrow \frac{1}{|I|}\int_I |S_{\mathcal{D}}^2 f(y) - \langle S_{\mathcal{D}}^2 f\rangle_I|\,dy = \frac{1}{|I|}\int_I \left|\sum_{J\subset I}(f,h_J)^2\frac{\mathbb{1}_J(y)}{|J|} - \frac{1}{|I|}\sum_{J\subset I}(f,h_J)^2\right|dy$$

$$\leq \frac{1}{|I|}\int_I \sum_{J\subset I}(f,h_J)^2 \frac{\mathbb{1}_J(y)}{|J|}\,dy + \frac{1}{|I|}\sum_{J\subset I}(f,h_J)^2$$

$$= \frac{2}{|I|}\sum_{J\subset I}(f,h_J)^2 \longrightarrow \boxed{\begin{gathered}\mathbb{1}_I(f-\langle f\rangle_I) = \sum_{J\subset I}(f,h_J)h_J \\ \int_I |f-\langle f\rangle_I|^2 = \sum_{J\subset I}(f,h_J)^2\end{gathered}}$$

$$= \frac{2}{|I|}\int_I |f-\langle f\rangle_I|^2$$

$$\Rightarrow (S_{\mathcal{D}}^2 f)^{\#}(x) \leq 2\left(f^{\#,2}(x)\right)^2 \quad /\!/$$

■

Remark: We can also use the Sharp Function to prove directly the lower bound $\|f\|_p \lesssim \|S_{\mathcal{D}} f\|_p$ for $p>2$:

$$(f_{\mathcal{D}}^{\#,2}(x))^2 = \sup_{I\ni x}\left(\frac{1}{|I|}\int_I |f(y)-\langle f\rangle_I|^2 dy\right) = \sup_{I\ni x}\left(\frac{1}{|I|}\sum_{J\subset I}(f,h_J)^2\right) = \sup_{I\ni x}\left(\frac{1}{|I|}\int_I \sum_{J\subset I}(f,h_J)^2\frac{\mathbb{1}_J(y)}{|J|}\,dy\right)$$

$$\leq \sup_{I\ni x}\left(\frac{1}{|I|}\int_I S_{\mathcal{D}}^2 f(y)\,dy\right)$$

$$= M_{\mathcal{D}}(S_{\mathcal{D}}^2 f)(x)$$

$$\Rightarrow \boxed{f_{\mathcal{D}}^{\#,2}(x) \leq \left(M_{\mathcal{D}}(S_{\mathcal{D}}^2 f)(x)\right)^{1/2}}$$

$$\Rightarrow \|f\|_p \underset{(1<p<\infty)}{\lesssim} \|f_{\mathcal{D}}^{\#,2}\|_p \leq \|(M_{\mathcal{D}}(S_{\mathcal{D}}^2 f))^{1/2}\|_p = \|M_{\mathcal{D}}(S_{\mathcal{D}}^2 f)\|_{p/2}^{1/2} \underset{p>2}{\lesssim} \|S_{\mathcal{D}}^2 f\|_{p/2}^{1/2} = \|S_{\mathcal{D}} f\|_p \quad /\!/$$

■